P425/1 PURE MATHEMATICS UNEB 2024

WORKED MARKING GUIDE SOLUTIONS

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SECTION A

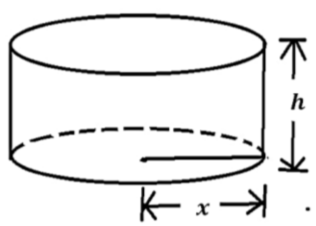
NO. 1

$4\,men$	$6\,women$
2	5
3	4
4	3

Number of ways

$$= {}^4C_2\,{}^6C_5 + {}^4C_3\,{}^6C_4 + {}^4C_4\,{}^6C_3$$
$$= 116\,ways$$

NO. 2



$$V = 1000cm^3$$
$$V = \pi x^2 h$$
$$1000 = \pi x^2 h$$
$$h = \frac{1000}{\pi x^2}$$

$S.A = \pi x^2 + 2\pi x h$ (area of circular base and curved surface)

$$A = \pi x^2 + 2\pi x\left(\frac{1000}{\pi x^2}\right)$$
$$A = \pi x^2 + 2000x^{-1}$$
$$\frac{dA}{dx} = 2\pi x - 2000x^{-2}, \frac{dA}{dx} = 0\ (at\ min\ of\ A)$$
$$\Rightarrow 0 = 2\pi x - \frac{2000}{x^2}$$
$$x^3 = \frac{1000}{\pi}, \quad x = \frac{10}{\sqrt[3]{\pi}}cm$$

NO. 3 a).

$$4x^2 + 25y^2 + 8x - 100y + 4 = 0$$
$$4x^2 + 8x + 25y^2 - 100y + 4 = 0$$
$$4(x^2 + 2x) + 25(y^2 - 4y) + 4 = 0$$
$$4((x+1)^2 - 1) + 25((y-2)^2 - 4) + 4 = 0$$
$$4(x+1)^2 - 1 + 25(y-2)^2 - 100 + 4 = 0$$
$$4(x+1)^2 + 25(y-2)^2 = 100$$
$$\frac{(x+1)^2}{25} + \frac{(y-2)^2}{4} = 1$$

$Centre\ is\ (-1, 2).$

NO. 3 b). $Usimg: b^2 = a^2(1 - e^2)$

$where\ \ a^2 = 25, b^2 = 4, \quad 0 < e < 1$

$$\Rightarrow 4 = 25(1 - e^2)$$
$$e^2 = 1 - \frac{4}{25} = \frac{21}{25} \ \Rightarrow\ e = \frac{\sqrt{21}}{5}$$

$The\ eccentricty\ of\ the\ ellipse\ is\ \frac{\sqrt{21}}{5}.$

NO. 4

$$Let\ \frac{1}{9 - x^2} = \frac{1}{(3-x)(3+x)}\ and$$
$$\frac{1}{(3-x)(3+x)} \equiv \frac{A}{3-x} + \frac{B}{3+x}$$
$$\Rightarrow 1 \equiv A(3+x) + B(3-x)$$

$When\ x = 3$	$When\ x = -3$
$6A = 1$	$6B = 1$
$A = \frac{1}{6}$	$B = \frac{1}{6}$

$$\therefore \frac{1}{(3-x)(3+x)} = \frac{1}{6(3-x)} + \frac{1}{6(3+x)}$$
$$\int_0^1 \frac{1}{9-x^2}dx = \int_0^1 \left[\frac{1}{6(3-x)} + \frac{1}{6(3+x)}\right]dx$$
$$= \frac{-1}{6}\ln(3-x) + \frac{1}{6}\ln(3+x)\Big|_0^1$$
$$= \frac{1}{6}\ln\left(\frac{3+x}{3-x}\right)\Big|_0^1$$
$$= \frac{1}{6}\ln 2 - \frac{1}{6}\ln 1$$
$$= \frac{1}{6}\ln 2\ \ as\ required.$$

NO. 5

Let x_0 be the initial population or first term of the G.P.

$$Common\ ratio, r = \frac{100 + 2.75}{100} = 1.0275$$
$$n^{th}\ term = ar^{n-1}$$
$$\Rightarrow 2x_0 = x_0(1.0275)^{n-1}$$
$$2 = \frac{1.0275^n}{1.0275}$$
$$2.055 = 1.0275^n$$
$$n = \frac{\ln(2.055)}{\ln(1.0275)}$$
$$n = 26.55035617 \approx 27\ years.$$

NO. 6

Taking L.H.S:

$$L.H.S = \frac{1-\cos 2x + 2\sin x \cos^2 x}{1+\cos 2x}$$

$$= \frac{1-(1-2\sin^2 x) + 2\sin x \cos^2 x}{1+(2\cos^2 x - 1)}$$

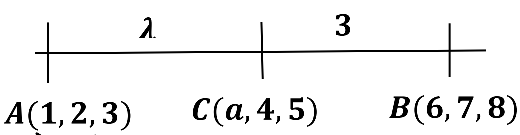
$$= \frac{2\sin^2 x + 2\sin x \cos^2 x}{2\cos^2 x}$$

$$= \frac{2\sin^2 x}{2\cos^2 x} + \frac{2\sin x \cos^2 x}{2\cos^2 x}$$

$$= \tan^2 x + \sin x$$

$$= \sin x + \tan^2 x \quad as\ required.$$

NO. 7



$\overrightarrow{AC} : \overrightarrow{CB} = \lambda : 3$

$$\frac{\overline{AC}}{\overline{CB}} = \frac{\lambda}{3} \Rightarrow 3\overline{AC} = \lambda\,\overline{CB}$$

$$3\left(\begin{pmatrix}a\\4\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix}\right) = \lambda\left(\begin{pmatrix}6\\7\\8\end{pmatrix} - \begin{pmatrix}a\\4\\5\end{pmatrix}\right)$$

$$3\begin{pmatrix}a-1\\2\\2\end{pmatrix} = \lambda\begin{pmatrix}6-a\\3\\3\end{pmatrix}$$

$$\begin{pmatrix}3a-3\\6\\6\end{pmatrix} = \begin{pmatrix}\lambda(6-a)\\3\lambda\\3\lambda\end{pmatrix}$$

$\Rightarrow 3\lambda = 6\ , \quad \lambda = 2$

$3a - 3 = \lambda(6-a)$

$3a - 3 = 2(6-a)$

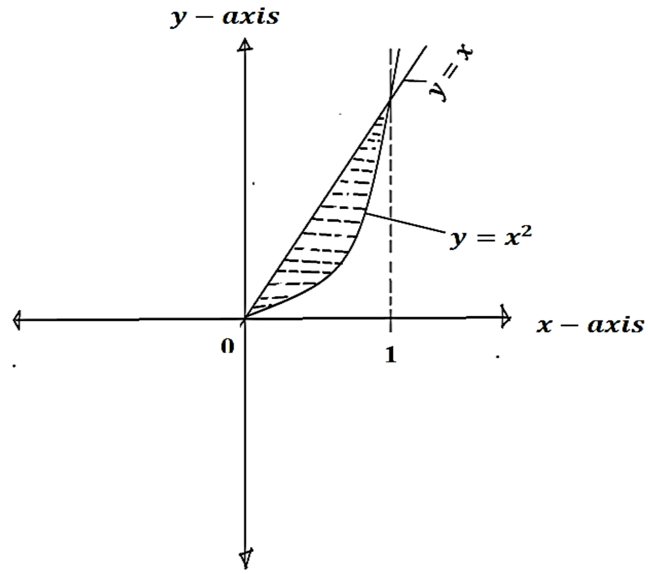
$3a - 3 = 12 - 2a$

$5a = 15$

$a = 3$

$\therefore a = 3\ and\ \lambda = 2.$

NO. 8



$$Area = \int_0^1 (x - x^2)dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1$$

$$= \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right)$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}\ square\ units.$$

SECTION B

NO. 9 a).

$$Let\ 12\cos\theta + 16\sin\theta \equiv R\cos(\theta - \alpha)$$

$$\Rightarrow 12\cos\theta + 16\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

$$\Rightarrow 12 = R\cos\alpha \ldots\ldots\ldots\ldots (i)$$

$$16 = R\sin\alpha \ldots\ldots\ldots\ldots (ii)$$

$$eqn\ (i)^2 + eqn\ (ii)^2$$

$$\Rightarrow (R\cos\alpha)^2 + (R\sin\alpha)^2 = 12^2 + 16^2$$

$$R^2 = 400$$

$$R = \pm 20$$

$$R = 20\ eqn\ (ii) \div (i)$$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{16}{12} \Rightarrow \tan\alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right) \Rightarrow \alpha = 53.13°$$

$$\therefore 12\cos\theta + 16\sin\theta = 20\cos(\theta - 53.13°)$$

NO. 9 b)(i) $At\ maximum\ \cos(\theta - 53.13°) = 1\ \ (12\cos\theta + 16\sin\theta)_{max} = 20(1) = 20$

$At\ minimum\ \cos(\theta - 53.13°) = -1$

$$(12\cos\theta + 16\sin\theta)_{min} = 20(-1) = -20$$

NO. 9 b)(ii) $12\cos\theta + 16\sin\theta = 15$

$$\Rightarrow \quad 20\cos(\theta - 53.13°) = 15$$
$$\cos(\theta - 53.13°) = \frac{15}{20}$$
$$\cos(\theta - 53.13°) = \frac{3}{4}$$
$$\theta - 53.13° = \cos^{-1}\left(\frac{3}{4}\right)$$
$$\theta = 41.41° + 53.13°$$
$$\therefore \theta = 94.54°$$

NO. 10 a)

$$Let\ f(x) = x^3 - 13x + p$$
$$x^3 - 13x + p = (x-4)q(x) + 0$$
$$when\ x = 4, \qquad 4^3 - 13(4) + p = 0$$
$$64 - 52 + p = 0$$
$$12 + p = 0 \ \Rightarrow p = -12$$
$$Then\ \ x^3 - 13x + p = 0$$
$$\Rightarrow x^3 - 13x - 12 = 0$$

$$\begin{array}{r|l} & x^2 + 4x + 3 \\ \hline x - 4 & x^3 - 13x - 12 \\ - & x^3 - 4x^2 \\ \hline & 4x^2 - 13x - 12 \\ - & 4x^2 - 16x \\ \hline & 3x - 12 \\ - & 3x - 12 \\ \hline & \ldots \end{array}$$

$$\therefore\ x^3 - 13x - 12 = (x-4)(x^2 + 4x + 3) = 0$$
$$Therefore\ \ x^2 + 4x + 3 = 0$$
$$(x+3)(x+1) = 0$$
$$either\ x = -3\ or\ x = -1.$$

NO. 10 b)

$$\frac{x^2 - x - 18}{x+3} \geq \frac{x}{2}$$
$$\frac{x^2 - x - 18}{x+3} - \frac{x}{2} \geq 0$$
$$\frac{2x^2 - 2x - 36 - x(x+3)}{2(x+3)} \geq 0$$
$$\frac{x^2 - 5x - 36}{2(x+3)} \geq 0$$
$$Critical\ values: x^2 - 5x - 36 = 0$$
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-36)}}{2(1)}$$
$$x = \frac{5 \pm 3}{2}$$
$$either\ x = 9\ or\ x = -4$$
$$Also, x + 3 = 0\ x = -3$$

	$x < -4$	$-4 < x < -3$	$-3 < x < 9$	$x > 9$
$\frac{x^2 - 5x - 36}{x+3}$	$+$	$+$	$-$	$+$

$$x \leq -4, \qquad -4 \leq x \leq -3, \ \ x \geq 9$$

NO. 11 a)

$$x = \sin\theta$$
$$dx = \cos d\theta$$
$$\theta = \sin^{-1}(x)$$

x	0	1
θ	0	$\pi/2$

$$\int_0^1 \frac{1+x}{\sqrt{1-x^2}}dx = \int_0^{\pi/2} \frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta d\theta$$
$$= \int_0^{\pi/2} (1+\sin\theta)d\theta$$
$$= \theta - \cos\theta \Big|_0^{\pi/2}$$
$$= \left(\frac{\pi}{2} - \cos\frac{\pi}{2}\right) - (0 - \cos 0)$$
$$= \frac{\pi}{2} + 1$$
$$= \frac{2+\pi}{2}$$

NO. 11 b).

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$
$$\sin y = \frac{x}{\sqrt{1+x^2}}$$
$$\frac{dy}{dx}\cos y = \frac{\sqrt{1+x^2} - x \cdot \frac{1}{2}(1+x^2)^{\frac{-1}{2}}(2x)}{\left(\sqrt{1+x^2}\right)^2}$$
$$\frac{dy}{dx}\cos y = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{(1+x^2)}$$

$$\frac{dy}{dx}=\frac{1+x^2-x^2}{(1+x^2)^{\frac{3}{2}}\cos y}=\frac{1}{(1+x^2)^{\frac{3}{2}}\sqrt{1-\sin^2 y}}$$

$$\frac{dy}{dx}=\frac{1}{(1+x^2)^{\frac{3}{2}}\left(1-\frac{x^2}{1+x^2}\right)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}=\frac{1}{(1+x^2)^{\frac{3}{2}}\left(\frac{1}{(1+x^2)^{\frac{1}{2}}}\right)}$$

$$\frac{dy}{dx}=\frac{1}{1+x^2}$$

NO. 12 a)

$x=-t^3+t^2+1$

$\frac{dx}{dt}=-3t^2+2t$

$y=t^2$

$\frac{dy}{dt}=2t$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$=(2t)\left(\frac{1}{-3t^2+2t}\right)$$

$$=\frac{2}{2-3t}$$

$m_1=\frac{2}{2-3t}$, $m_2=?$

$y=\frac{2}{3}x+\frac{1}{3}$

$m_2=\frac{2}{3}$

$m_1=m_2$ *(parallel lines)*

$$\Rightarrow \frac{2}{2-3t}=\frac{2}{3} \Rightarrow 6=4-6t$$

$$6t=-2$$

$$t=-\frac{1}{3}$$

$$x=-\left(-\frac{1}{3}\right)^3+\left(-\frac{1}{3}\right)^2+1$$

$$x=\frac{31}{27}$$

$$y=\left(-\frac{1}{3}\right)^2=\frac{1}{9}$$

$B\left(\frac{31}{27},\frac{1}{9}\right)$

b) $y=mx+c$

$$B\left(\frac{31}{27},\frac{1}{9}\right),\qquad m=\frac{2}{3}$$

$$\Rightarrow \frac{1}{9}=\frac{2}{3}\left(\frac{31}{27}\right)+c$$

$$c=\frac{1}{9}-\frac{62}{81}$$

$$c=\frac{9-62}{81}$$

$$c=-\frac{51}{84}$$

$$y=\frac{2}{3}x-\frac{51}{81}$$

NO. 13 a). *Let* $f(x)=\ln(1-2x)$

$$f(0)=\ln(1-2(0))=0$$

$$f'(x)=\frac{-2}{1-2x}=-2(1-2x)^{-1}$$

$$f'(0)=-2$$

$$f''(x)=2(1-2x)^{-2}(-2)$$

$$f''(x)=-4(1-2x)^{-2}$$

$$f''(0)=-4$$

$$f'''(x)=8(1-2x)^{-3}(-2)$$

$$f'''(0)=-16$$

$$Then\ \ f(x)=f(0)+\frac{f'(0)x}{1!}+\frac{f''(0)}{2!}+\cdots$$

$$\ln(1-2x)=0-2x-\frac{4x^2}{2}-\frac{16x^3}{3!}+\cdots$$

$$=-2x-2x^2-\frac{8x^3}{3}+\cdots$$

$$\therefore \ln(1-2x)=-2x-2x^2-\frac{8x^2}{3}+\cdots$$

NO. 13 b) $\tan 46°$

Let $y=\tan x\ \ y+\delta y=\tan(x+\delta x)$

where δy *and* δx *are small changes in* y *and* x *respectively.*

Let $x+\delta x=46°$ *and* $x=45°$

$\Rightarrow\ \delta x=46°-45°=1°$

$\delta x=\frac{1}{180}\pi=\frac{\pi}{180}\ \ \frac{dy}{dx}=\sec^2 x=1+\tan^2 x$

$\frac{dy}{dx}=1+\tan^2(45°)=2$

as $\delta x\to 0,\delta y\to 0.$ *Then* $\delta y\approx\frac{dy}{dx}\cdot\delta x$

$\delta y=2\left(\frac{\pi}{180}\right)=\frac{\pi}{90}$

$y = \tan 45°$
$y = 1$
$Using\ y + \delta y = \tan(x + \delta x)$
$\Rightarrow 1 + \frac{\pi}{90} = \tan 46°$
$\tan 46° = \frac{90+\pi}{90}$ $\tan 46° = 1.034906585$
$\therefore \tan 46° = 1.035\ (3dps).$

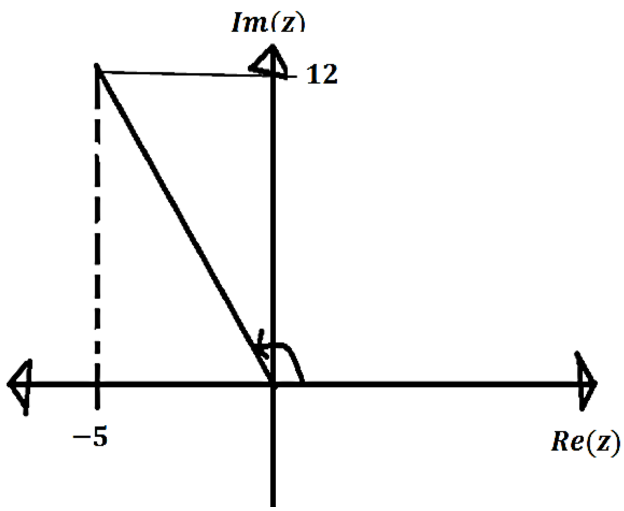
NO. 14a) $Let\ \ z = x + iy$

$$3|z-2| = |z-6i|$$
$$3|x+iy-2| = |x+iy-6i|$$
$$3\sqrt{(x-2)^2+y^2} = \sqrt{x^2+(y-6)^2}$$
$$9[(x-2)^2+y^2] = x^2+(y-6)^2$$
$$9(x^2-4x+4+y^2) = x^2+y^2-12x+36$$
$$9x^2-36x+36+9y^2 = x^2+y^2-12y+36$$
$$8x^2+8y^2-36x+12y = 0$$
$$x^2+y^2-\frac{36}{8}x+\frac{12}{8}y = 0$$
$$x^2+y^2-\frac{9}{2}x+\frac{3}{2}y = 0$$

which is an equation of a circle since it satisfies the form $x^2+y^2+2gx+2fy+c = 0.$

NO. 14b) $z = -5+12i$

$$r = |z| = \sqrt{(-5)^2+12^2}$$
$$r = \sqrt{169}\ \Rightarrow\ r = 13$$



$$Arg(z) = \pi - \tan^{-1}\left(\frac{12}{5}\right) = 0.6257\pi$$
$$\arg(z) = 0.6257\pi + 2\pi k, k \in \mathbb{Z}$$
$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

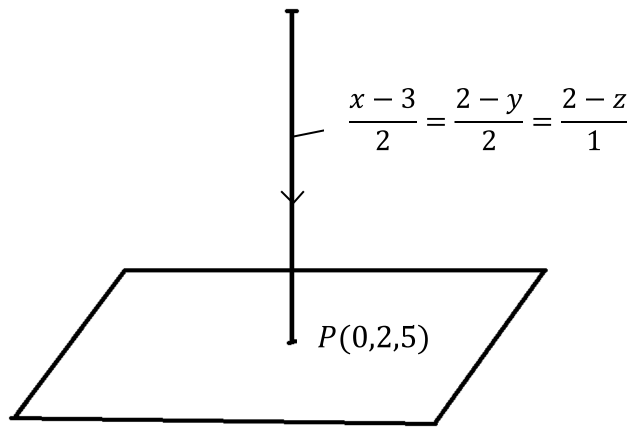
$$z^{\frac{1}{2}} = 13^{\frac{1}{2}}\left[\cos\left(\frac{0.6257\pi+2\pi k}{2}\right) + i\sin\left(\frac{0.6257\pi+2\pi k}{2}\right)\right]$$
$$for\ k = 0,1$$

$When\ k = 1$

$$z^{\frac{1}{2}} = 13^{\frac{1}{2}}\left[\cos\left(\frac{0.6257\pi+2\pi}{2}\right) + i\sin\left(\frac{0.6257\pi+2\pi}{2}\right)\right]$$
$$z^{\frac{1}{2}} = -2-3i$$
$$z^{\frac{1}{2}} = -(2+3i)$$
$$z^{\frac{1}{2}} = \pm(2+3i)$$

NO. 15 a)

$$P(0,1,5), Q(-1,3,1)$$
$$\frac{x-3}{2} = \frac{2-y}{2} = \frac{2-z}{1}$$



$Let\ \ \tilde{n} = \begin{pmatrix}2\\-2\\-1\end{pmatrix}$ $\tilde{a} = \begin{pmatrix}0\\2\\5\end{pmatrix}$ $\tilde{n}\cdot\tilde{r}$

$$= \tilde{n}\cdot\tilde{a}\ \begin{pmatrix}2\\-2\\-1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}$$
$$= \begin{pmatrix}2\\-2\\-1\end{pmatrix}\begin{pmatrix}0\\2\\5\end{pmatrix}\ 2x-2y-z$$
$$= 0-4-5\ 2x-2y-z+9 = 0$$

b) $Q(-1,3,1)$

$$2(-1)-2(3)-1+9 = 0$$
$$-2-6-1+9 = 0$$
$$-9+9 = 0$$
$$0 = 0$$

Since $L.H.S = R.H.S$, Then Q lies on the plane.

NO. 15 c)

$$Let\ \frac{x-3}{2}=\frac{y-2}{-2}=\frac{z-2}{-1}=\lambda$$

Then

$$\frac{x-3}{2}=\lambda \qquad \frac{y-2}{2}=\lambda \qquad \frac{z-2}{-1}=\lambda$$

$$x=2\lambda+3 \qquad y=-2\lambda+2 \qquad z=-\lambda+2$$

$$\Rightarrow 2(2\lambda+3)-2(-2\lambda+2)-(-\lambda+2)+9=0$$

$$4\lambda+6+4\lambda-4+\lambda-2+9=0$$

$$9\lambda=-9$$

$$\lambda=-1$$

$x=2(-1)+3 \quad \Rightarrow\ x=1$

$y=-2(-1)+2 \quad \Rightarrow y=4$

$$z=-(-1)+2 \quad \Rightarrow z=3$$

$$R(1,4,3)$$

d) $\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}=\begin{pmatrix}1\\4\\3\end{pmatrix}-\begin{pmatrix}0\\2\\5\end{pmatrix}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$

$\overrightarrow{QR}=\overrightarrow{OR}-\overrightarrow{OQ}=\begin{pmatrix}1\\4\\3\end{pmatrix}-\begin{pmatrix}-1\\3\\1\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}$

$\overrightarrow{PR}\cdot\overrightarrow{QR}=\begin{pmatrix}1\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix}=2+2-4=0$

Since $\overrightarrow{PR}\cdot\overrightarrow{QR}=0$, then $\overrightarrow{PR}$ is perpendicular to $\overrightarrow{QR}$.

NO. 16 a)

$\frac{dM}{dt}\propto(M_0-M)$

$\frac{dM}{dt}=K(M_0-M)$

$\frac{dM}{dt}=K(10-M)$

where K is a constant.

NO. 16 b(i)

$\frac{dM}{dt}=K(10-M)$

$\frac{1}{10-M}dM=Kdt$

$\int\frac{1}{10-M}dM=\int Kdt$

$-\ln(10-M)=Kt+C$

when $t=0, M=0$

$\Rightarrow -\ln(10-0)=C$

$C=-\ln 10$

$-\ln(10-M)=Kt-\ln 10$

when $t=1, M=2$

$\Rightarrow -\ln(10-2)=K-\ln 10$

$K=\ln 10-\ln 8$

$K=\ln\left(\frac{5}{4}\right)$

$-\ln(10-M)=t\ln\left(\frac{5}{4}\right)-\ln 10$

$\ln\left(\frac{10}{10-M}\right)=\ln\left(\frac{5}{4}\right)^t$

$\Rightarrow \frac{10}{10-M}=\left(\frac{5}{4}\right)^t$

$\frac{10-M}{10}=\left(\frac{4}{5}\right)^t, M=10-10\left(\frac{4}{5}\right)^t$

NO. 16 b(ii)

$M=10-10\left(\frac{4}{5}\right)^5$

$M=6.7232$

$M=7\ years.$
