UACE SUBSIDIARY MATHEMATICS

HOME SCHOOL SELF STUDY NOTES

QUADRATIC EQUATIONS

Any equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation and the values of x, which satisfy the equation, are called roots.

Solution of a quadratic equation that factorizes

Example

1. Find the roots of the equation $x^2 - 5x + 6 = 0$

Solution

$x^2 - 2x - 3x + 6 = 0$

$x(x - 2) - 3(x - 2) = 0$

$(x - 2)(x - 3) = 0$ Either $x - 2 = 0, x = 2$ $or\ x - 3 = 0, x = 3$

Solution of a quadratic equation that does not factorize

By completing the square

This method uses the expansion $(x + b)^2 = x^2 + 2bx + b^2$. It is important to note that the last term b^2, is the square of half the coefficient of x, $(2b)$.

Examples

1. Find the roots of the equation $2x^2 - 5x + 1 = 0$

Solution

Dividing through by 2 gives;

$$x^2 - \frac{5}{2}x + \frac{1}{2} = 0$$

$$x^2 - \frac{5}{2}x = -\frac{1}{2}$$

Adding the square of half the coefficient of x to both sides of the equation;

$$x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = -\frac{1}{2} + \left(\frac{5}{4}\right)^2$$

$$\left(x - \frac{5}{4}\right)^2 = -\frac{1}{2} + \frac{25}{16}$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{17}{16}$$

$$\sqrt{\left(x - \frac{5}{4}\right)^2} = \sqrt{\frac{17}{16}}$$

$$x - \frac{5}{4} = \frac{\sqrt{17}}{4}$$

$$x = \frac{5 \pm \sqrt{17}}{4} \qquad \therefore x = 2.281 \quad Or\ x = 0.219$$

2. Solve $2x^2 - 6x + 4 = 0$

Solution

$2x^2 - 6x + 4 = 0$

$x^2 - 3x + 2 = 0$

$x^2 - 3x = -2$

Adding the square of half the coefficient of x to each side of the equation

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = -2 + \left(\frac{3}{2}\right)^2$$

$$\left(x - \frac{3}{2}\right)^2 = -2 + \frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{1}{4}$$

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \sqrt{\frac{1}{4}}$$

$$x - \frac{3}{2} = \pm\frac{1}{2}$$

$$x = \frac{3 \pm 1}{2} \qquad x = 2 \; or \; x = 1$$

3. Solve $x^2 + 3x - 1 = 0$

<u>Solution</u>

$x^2 + 3x = 1$

Adding the square of half the coefficient of x to each side of the equation gives;

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = 1 + \left(\frac{3}{2}\right)^2$$

$$(x + 3)^2 = \frac{13}{4}$$

$$x + \frac{3}{2} = \pm\frac{\sqrt{13}}{2} \text{ giving } x = \frac{\sqrt{13} - 3}{2} \quad or \; x = \frac{-\sqrt{13} - 3}{2}$$

$$x = 0.30 \quad or \quad -3.30$$

Note: The method of completing the square, used to solve $ax^2 + bx + c = 0$ can also be used to find the maximum of minimum value of the expression $ax^2 + bx + c$.

For example, consider the expression $x^2 + 3x + 4$

$$x^2 + 3x + 4 = x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4$$

$$= \left(x + \frac{3}{2}\right)^2 + \frac{7}{4}$$

Now $\left(x + \frac{3}{2}\right)^2$ cannot be negative for any value of x, i.e. $\left(x + \frac{3}{2}\right)^2 \geq 0$

Thus $x^2 + 3x + 4$ is always positive and will have a minimum value of $\frac{7}{4}$ when $x + \frac{3}{2} = 0$,ie when $x = -\frac{3}{2}$

<u>Example</u>

Find the maximum value of $5 - 2x - 4x^2$

<u>Solution</u>

Let's first rewrite $5 - 2x - 4x^2 = -4x^2 - 2x + 5$

$$= -4\left(x^2 + \frac{1}{2}x\right) + 5$$

$$= -4\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \frac{4}{16} + 5$$

$$= -4\left(x + \frac{1}{4}\right)^2 + \frac{21}{4}$$

$$= \frac{21}{4} - 4\left(x + \frac{1}{4}\right)^2$$

Now $\left(x + \frac{1}{4}\right)^2 \geq 0$

Thus $5 - 2x - 4x^2$ has a maximum value of $\frac{21}{4}$ when $x = -\frac{1}{4}$

IN GENERAL

For a quadratic equation $ax^2 + bx + c = 0$, the roots can be obtained from the formula;

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Solve $x^2 + 3x - 1 = 0$

Solution

Comparing with the general equation $ax^2 + bx + c = 0$ $a = 1, b = 3, c = -1$

Substituting in the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 - 1}}{2 \times 1}$$

$$x = \frac{-3 \pm \sqrt{9 + 4}}{2}$$

$x = \frac{-3 + \sqrt{13}}{2}$ Or $x = \frac{-3 - \sqrt{13}}{2}$ $\therefore x = 0.30, x = -3.30$

ROOTS OF QUADRATIC EQUATIONS

If the equation $ax^2 + bx + c = 0$ has roots α and β, then its equivalent equation will be;

$(x - \alpha)(x - \beta) = 0$ $, as\ it\ gives\ x = \alpha\ or\ x = \beta$

$x^2 - \beta x - \alpha x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}$

$x^2 - (\alpha + \beta)x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}$

By comparing the coefficients on both sides, we obtain

$\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Hence the equation $ax^2 + bx + c = 0$ can be written in the form;

$x^2 - (Sum\ of\ roots)x + (product\ of\ roots) = 0$

Example

1. Write down the sum and product of the roots of the following equations;

(i) $3x^2 - 2x - 7 = 0$ (ii) $5x^2 + 11x + 3 = 0$ (iii) $2x^2 + x - 7 = 0$

Solution

(i) $x^2 - \frac{2}{3}x - \frac{7}{3} = 0$;sum of roots $= -\left(-\frac{2}{3}\right) = \frac{2}{3}$ and product of roots $= -\frac{7}{3}$

(ii) $x^2 + \frac{11}{5}x + \frac{3}{5} = 0$; sum of roots $= -\frac{11}{5}$ and product of roots $= \frac{3}{5}$

(iii) $x^2 + \frac{1}{2}x - \frac{7}{2} = 0$; sum of roots $= \frac{1}{2}$ *and product of roots* $= -\frac{7}{2}$

2. Find the equation whose roots are $\frac{3}{4}$ *and* $-\frac{1}{2}$

Solution

Sum of roots $= \frac{3}{4} + \left(-\frac{1}{2}\right) = \frac{1}{4}$ and product of roots $= \frac{3}{4} \times \left(-\frac{1}{2}\right) = -\frac{3}{8}$

The equation is in the form $x^2 - (Sum\ of\ roots)x + (product\ of\ roots) = 0$

$$x^2 - \left(\frac{1}{4}\right)x + \left(\frac{-3}{8}\right) = 0$$

$$8x^2 - 2x - 3 = 0$$

3. Find the equations whose roots are $\frac{1}{3}$ and $-\frac{1}{4}$

Solution

Sum of roots $=\frac{1}{3}+-\frac{1}{4}=\frac{1}{3}-\frac{1}{4}=\frac{4-3}{12}=\frac{1}{12}$

Product of roots $=\frac{1}{3}\times-\frac{1}{4}=-\frac{1}{12}$

The equation is in the form $x^2-(Sum\ of\ roots)x+(product\ of\ roots)=0$

$$x^2-\left(\frac{1}{12}\right)x+\left(\frac{-1}{12}\right)=0$$

$$12x^2-x-1=0$$

4. The roots of the equation $3x^2+4x-5=0$ $are\ \alpha\ and\ \beta, find\ the\ values\ of;$

(i) $\frac{1}{\alpha}+\frac{1}{\beta}$ $(ii)\ \alpha^2+\beta^2$

Solution

$\alpha+\beta=-\frac{4}{3}$ $\alpha\beta=-\frac{5}{3}$

(i) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{\frac{-4}{3}}{\frac{-5}{3}}==-\frac{4}{3}\times-\frac{3}{5}=\frac{4}{5}$

(ii) From $(\alpha+\beta)^2=(\alpha+\beta)(\alpha+\beta)=\alpha^2+2\alpha\beta+\beta^2$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

$$=\left(\frac{-4}{3}\right)^2-2\left(\frac{-5}{3}\right)$$

$$=\frac{16}{9}+\frac{10}{3}=\frac{16+30}{9}=5\frac{1}{9}$$

5. The roots of the equation $2x^2-7x+4=0$ $are\ \alpha\ and\ \beta.Find\ the\ equation$ $whose\ roots\ are$ $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

Solution

From the given equation, sum of roots, $\alpha+\beta=\frac{7}{2}$ $and\ product\ of\ roots\ \alpha\beta=2$

For the new roots, sum $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{\left(\frac{7}{2}\right)^2-4}{2}=\frac{\left(\frac{49}{4}\right)-4}{2}=\frac{33}{8}$

Product of new roots, $\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=1$

The equation is given by $x^2-(Sum\ of\ roots)x+(product\ of\ roots)=0$

$$x^2-\frac{33}{8}x+1=0$$

$$8x^2-33x+8=0$$

6. Find the values of k if the roots of the equation $3x^2+5x-k=0$ $differ\ by\ 2$

Solution

Let one root be α $, then\ the\ other\ will\ be\ \alpha+2$

Sum of roots $\alpha+\alpha+2=-\frac{5}{3}$, $2\alpha=-\frac{5}{3}-2\ \Rightarrow\ \alpha=-\frac{11}{6}$

Product of roots $\alpha(\alpha+2)=-\frac{k}{3}$, $\alpha^2+2\alpha=-\frac{k}{3}$ ……………… ***

Substituting for $\alpha\ in\ equation$ $***$ gives;

$$\left(\frac{-11}{6}\right)^2+2\left(\frac{-11}{6}\right)=-\frac{k}{3}$$

$\frac{121}{36} - \frac{22}{6} = -\frac{k}{3}$

$\frac{121-132}{36} = -\frac{k}{3}$, $-\frac{11}{36} = -\frac{k}{3}$ $\therefore k = \frac{11}{12}$

7. If one of the roots of the equation $27x^2 + bx + 8 = 0$ is the square of the other, find b.

Solution

Let one root be α, *then the other will be* α^2, then;

Sum of roots $\alpha + \alpha^2 = -\frac{b}{27} \ldots\ldots(i)$ *and product of roots* $\alpha \times \alpha^2 = \frac{8}{27} \ldots\ldots\ldots(ii)$

$\alpha^3 = \left(\frac{2}{3}\right)^3$ *hence* $\alpha = \frac{2}{3}$ Which we substitute in equation (i) to find b;

$\frac{2}{3} + \left(\frac{2}{3}\right)^2 = -\frac{b}{27}$

$\frac{2}{3} + \frac{4}{9} = -\frac{b}{27}$

$\frac{10}{9} = -\frac{b}{27}$ $\therefore b = -30$

The discriminant

The value of the expression $(b^2 - 4ac)$ will determine the nature of the roots of the quadratic equation $ax^2 + bx + c = 0$ and it is called discriminant i.e. it discriminates between the roots of the equation.

For;

(i) Two real roots , $b^2 - 4ac > 0$

(ii) Repeated or equal roots $b^2 - 4ac = 0$

(iii) No real roots, $b^2 - 4ac < 0$

Example

Given that the equation

$(5a + 1)x^2 - 8ax + 3a = 0$ *has equal roots , find the possible values of a*

Solution

We identify a, b and c from the above equation and then apply the condition for equal roots

a = $(5a + 1)$, b=$-8a$ *and* $c = 3a$

For equal roots, $b^2 - 4ac$=0

$(-8a)^2 - 4(5a + 1)(3a)$=0

$64a^2 - 12a(5a + 1) = 0$

$64a^2 - 60a^2 - 12a = 0$

$4a^2 - 12a = 0$

$4a(a - 3) = 0$

Either 4a=0 or $a - 3 = 0$ $\therefore a = 0$ *or* $a = 3$

Trial questions

1. State (i) the sum (ii) the product of the roots of each of the following equations
   (a) $x^2 + 9x + 4 = 0$ (b) $x^2 - 7x + 2 = 0$ (c) $2x^2 - 7x + 1 = 0$ (d) $3x^2 + 10x - 2 = 0$
   [Ans: a) -9 , 4 (b) 2 , -5 (c) 7/2 , 1/2 (d) -10/3 , -2/3]

2. In each part of this question, you are given the sum and product of the roots of a quadratic. Find the quadratic equation in the form $ax^2 + bx + c = 0$

	a	b	c	D	e	f	g
sum	-3	6	7	-2/3	-5/2	-3/4	-1/4
Product	-1	-4	-5	-7/3	-2	-5	-1/3

[Ans: (a) $x^2 + 3x - 1 = 0$ (b) $x^2 - 6x - 4 = 0$ (c) $x^2 - 7x - 5 = 0$ (d) $3x^2 + 2x - 7 = 0$ (e) $x^2 + 5x - 4 = 0$ (f) $2x^2 + 3x - 10 = 0$ (g) $12x^2 + 3x - 4 = 0$]

3. If α, β are the roots of the equation $3x^2 - x - 1 = 0$,form the equations whose roots are;
(i) $2\alpha, 2\beta$ (ii) α^2, β^2 $(iii) \frac{1}{\alpha}, \frac{1}{\beta}$ $(iv) \alpha + 1, \beta + 1$

[Ans: (i) $3x^2 - 2x - 4 = 0$ $(ii) 9x^2 - 7x + 1 = 0$ (iii) $x^2 + x - 3 = 0$ $(iv) 3x^2 - 7x + 3 = 0$]

4. One of the roots of the equation $ax^2 + bx + c = 0$ is three times the other .Show that $3b^2 - 16ac = 0$

5. If the roots of the equation $2x^2 - 7x + 1 = 0$ $are\ \alpha\ and\ \beta$.find the quadratic equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ [Ans: $x^2 + 45x + 4 = 0$]

6. Given that $\alpha\ and\ \beta\ are\ the\ roots\ of\ the\ quadratic\ equation$ $3x^2 - x - 5 = 0$. Form the equation whose roots are $2\alpha - \frac{1}{\beta}$ and $2\beta - \frac{1}{\alpha}$ [Ans:$15x^2 - 13x - 169 = 0$]

7. One root of the equation $2x^2 - x + c = 0$ $is\ twice\ the\ other$ .Find the value of c [Ans:c=$\frac{1}{9}$]

8. Find the value of k for which the equation $4(x-1)(x-2) = k$ has roots which differ by 2 [Ans:k=3]

9. If the roots of the equation $x^2 + px + 7 = 0$ $are\ \alpha\ and\ \beta$. Find the possible values of p [Ans: p= ± 6]

10. Find the quadratic equation, which has the difference of its roots equal to 2 and the difference of the squares of its roots equal to 5. [Ans: $16x^2 - 40x + 9 = 0$]

11. Each of the following expressions has a maximum or minimum value for all real values Find (i) which it is, maximum or minimum, (ii) its value, (iii) the value of x

(a) $x^2 + 4x - 3$ [Ans: (i) min (ii) -7 (ii) -2]

(b) $2x^2 + 3x + 1$ [Ans: (i) min (ii) $-\frac{1}{8}$ (ii) $-\frac{3}{4}$]

(c) $x^2 - 6x + 1$ [Ans: (i) min (ii) -8 (ii) 3]

(d) $3 - 2x - x^2$ [Ans: (i) max (ii) 4 (ii) -1]

(e) $5 + 2x - x^2$ [Ans: (i) max (ii) 6 (ii) 1]

MATRICES

A matrix is a rectangular array of numbers called elements or entries. Information can conveniently be presented as an array of rows and columns.

Order of a matrix

The order of a matrix gives the format of how a matrix should be written. It is always in the form **m × n** where m is the number of rows and n is the number of columns in the matrix. For example

(i) A 2×2 matrix

In this matrix the number of rows is 2 and the columns are also 2 i.e.

$$\begin{pmatrix} 8 & 1 \\ -3 & 4 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(ii) A 3×3 matrix

In this matrix the number of rows is 3 and the columns are also 3 i.e.

$$\begin{pmatrix} 3 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 9 & 2 \end{pmatrix}, \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Note: Other matrices of different order are possible i.e. $1 \times 2, 2 \times 1, 1 \times 3, 3 \times 1, 2 \times 3, 3 \times 2, e.t.c.$

Operations on matrices

Addition and Subtraction

Two or more matrices can be added if they have the same order i.e. the number of rows and columns in the first matrix must be equal to the number of rows and columns in the second matrix.

Examples

1. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

2. $\begin{pmatrix} -2 & 0 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2+-1 & 0+3 \\ 3+0 & 2+2 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 3 & 4 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1+3 & 0+2 & 1+1 \\ 3+2 & -1+0 & 2+-3 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 5 & -1 & -1 \end{pmatrix}$

4. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$

5. $\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} -1 & -3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3--1 & 1--3 \\ -2-0 & 0-2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ -2 & -2 \end{pmatrix}$

6. $\begin{pmatrix} 6 & 3 \\ 1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 8 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 6-0 & 3--1 \\ 1-8 & 2-1 \\ 1-3 & 0-0 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ -7 & 3 \\ -2 & 0 \end{pmatrix}$

Multiplication of matrices

Scalar multiplication

This is the type of multiplication where we multiply a given matrix with a constant which is taken as a scalar.

Examples

1. Expand $a\begin{pmatrix} b & c \\ e & f \end{pmatrix}$

Solution

$a\begin{pmatrix} b & c \\ e & f \end{pmatrix} = \begin{pmatrix} a\times b & a\times c \\ a\times e & a\times f \end{pmatrix} = \begin{pmatrix} ab & ac \\ ae & af \end{pmatrix}$

2. Given matrix $A = \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ -2 & 8 \end{pmatrix}$

Find (i) 2A (ii) 4B – A (iii) 3(A+B)

Solution

(i) $2A = 2\begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2\times 3 & 2\times 0 \\ 2\times 1 & 2\times -2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 2 & -4 \end{pmatrix}$

(ii) $4B = 4\begin{pmatrix} 0 & 3 \\ -2 & 8 \end{pmatrix} = \begin{pmatrix} 4\times 0 & 4\times 3 \\ 4\times -2 & 4\times 8 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ -8 & 32 \end{pmatrix}$

$4B - A = \begin{pmatrix} 0 & 12 \\ -8 & 32 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 12 \\ -9 & 34 \end{pmatrix}$

(iii) $A + B = \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -2 & 8 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -1 & 6 \end{pmatrix}$

$3(A + B) = 3\begin{pmatrix} 3 & 3 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ -3 & 18 \end{pmatrix}$

General multiplication of matrices

We can multiply two or more matrices if and only if the number of columns in the first matrix are equal to the number of rows in the second matrix.

Examples

Expand

(i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

Solution

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a\times e + b\times g & a\times f + b\times h \\ c\times e + d\times g & c\times f + d\times h \end{pmatrix}$

Hence when we are expanding, we multiply row by column

(ii) $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3\times 0 + 1\times 3 & 3\times 1 + 1\times 1 \\ 2\times 0 + 1\times 3 & 2\times 1 + 1\times 1 \end{pmatrix} = \begin{pmatrix} 0+3 & 3+1 \\ 0+3 & 2+1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3 & 3 \end{pmatrix}$

(iii) $\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}^2 = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3\times 3 + 4\times 2 & 3\times 4 + 4\times 5 \\ 2\times 3 + 5\times 2 & 2\times 4 + 5\times 5 \end{pmatrix}$

$= \begin{pmatrix} 9+8 & 12+20 \\ 6+10 & 8+25 \end{pmatrix} = \begin{pmatrix} 17 & 32 \\ 16 & 33 \end{pmatrix}$

(iv) Multiply $\begin{pmatrix} 8 & 9 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 4 & 0 \end{pmatrix}$

Solution

$\begin{pmatrix} 8 & 9 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 8\times -2 + 9\times 4 & 8\times 3 + 9\times 0 \\ 5\times -2 + -1\times 4 & 5\times 3 + -1\times 0 \end{pmatrix}$

$$= \begin{pmatrix} -16+36 & 24+0 \\ -10+-4 & 15+0 \end{pmatrix} = \begin{pmatrix} 20 & 24 \\ -14 & 15 \end{pmatrix}$$

(v) Given the matrices below;

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Their matrix product is;

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

yet **BA** is not defined.

(vi). Given the matrices below;

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Their matrix products are:

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1\times a+2\times c & 1\times b+2\times d \\ 3\times a+4\times c & 3\times b+4\times d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} a\times 1+b\times 3 & a\times 2+b\times 4 \\ c\times 1+d\times 3 & c\times 2+d\times 4 \end{pmatrix} = \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}.$$

Note: In general, when multiplying matrices, the commutative law doesn't hold, i.e. $AB \neq BA$ as seen in the above example.

The determinant of a matrix

Consider a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is denoted by det A where det A $= ad - bc$. The matrix which has a determinant of zero is called a singular matrix

Examples

1. If M $= \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$, Find det M

Solution

Det M $=(4\times -1) - (1\times 3) = -4 - 3 = -7$

2. $If\ A = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$, find det A

solution

Det A $= (1\times 0) - (3\times 1) = 0 - 3 = -3$

3. Given that $A = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$, Find (i) det (3A + B) (ii) det (2A – B)

Solution

(i) $3A + B = 3\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ 4 & 1 \end{pmatrix}$

Det $(3A + B) = (6 \times 1) - (11 \times 4) = 6 - 44 = -38$

(ii) $(2A - B) = 2\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$

Given that $A = \begin{pmatrix} 4 & 6 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -4 \\ 2 & 1 \end{pmatrix}$. Show that A + B is a singular matrix.

Solution

$A + B = \begin{pmatrix} 4 & 6 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$

Det $(A + B) = 3 \times 2 - 2 \times 3 = 6 - 6 = 0$

Since det (A +B) = 0, A + B is a singular matrix

Formation of a matrix

When forming matrices, we consider the number of rows as well as the number of columns required for a certain matrix.

Examples

(i) A 3×1 matrix

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

(ii) A 2×2 matrix

$$\begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix}$$

(iii) A 4×3 matrix

$$\begin{pmatrix} 6 & 0 & 7 \\ 2 & 1 & 2 \\ 4 & 5 & 8 \\ -1 & 9 & 1 \end{pmatrix}$$

Inverse of a matrix

The inverse of a matrix A is given by $\frac{1}{\det A} \times \textit{the adjoint matrix}$. The inverse of a matrix A is denoted by A^{-1}. To get the adjoint, we interchange the entries of the major diagonal and multiply the entries of the minor diagonal by -1 i.e.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $Adjoint\ A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Det A $= ad - bc$

$$A^{-1} = \frac{1}{\det A} \times Adjoint\ A$$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Note: The inverse of a singular matrix does not exist because we end up with a division by zero which is undefined.

Examples

If $A = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$, find (i) A^{-1} (ii) B^{-1} (iii) $(A+B)^{-1}$

Solution

(i) Det A $= (3 \times 1) - (1 \times 0) = 3$

Adjoint A $= \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$$

(ii) Det B $= (-1 \times 3) - (2 \times 1) = -3 - 2 = -5$

Adjoint B $= \begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix}$

$$B^{-1} = \frac{1}{-5}\begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix}$$

(iii) $A + B = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

Det $(A + B) = (2 \times 4) - (3 \times 1) = 8 - 3 = 5$

Adjoint $(A + B) = \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$

$$(A+B)^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$$

Note: $AA^{-1} = I$ where I is an identity matrix where an identity matrix which has the entries in the major diagonal equal to one and the entries in the minor diagonal all equal to zero e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *is* 2×2 identity matrix.

Solving simultaneous equations using matrices

One of the most important applications of matrices is to find the solution of linear simultaneous equations. It is a requirement to first re-arrange the given simultaneous equations into matrix format.

Example 1

Consider the simultaneous equations

$x + 2y = 4$

$3x - 5y = 1$

Provided you understand how matrices are multiplied together you will realise that these can be written in matrix form as;

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Writing

$A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$,

We have AX = B

This is the **matrix form** of the simultaneous equations. Here the unknown is the matrix X, since A and B are already known. A is called the **matrix of coefficients.**

Now given AX = B, we can multiply both sides by the inverse of A, provided this exists, to give;

$$A^{-1}AX = A^{-1}B$$

But $AA^{-1} = I$, the identity matrix. Furthermore, IX = X, because multiplying any matrix by an identity matrix of the appropriate size leaves the matrix unaltered. So $X = A^{-1}B$

Thus if AX = B then $X = A^{-1}B$

This result gives us a method for solving simultaneous equations. All we need do is write them in matrix form, calculate the inverse of the matrix of coefficients, and finally perform a matrix multiplication.

<u>Solution to the above question</u>

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

We need to calculate the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$

Det A = (1 × −5) − (2 × 3) = −11

$$A^{-1} = -\frac{1}{11}\begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix}$$

$$X = A^{-1}B = -\frac{1}{11}\begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= -\frac{1}{11}\begin{pmatrix} -5 \times 4 + -2 \times 1 \\ -3 \times 4 + 1 \times 1 \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} -22 \\ -11 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow x = 2 \; and \; y = 1$$

<u>Example 2:</u>

Using matrices, calculate the values of x and y for the following simultaneous equations:

$2x - 2y - 3 = 0$

$8y = 7x + 2$

<u>Solution:</u>

Step 1: Write the equations in the form $ax + by = c$

$2x - 2y - 3 = 0 \Rightarrow 2x - 2y = 3$

$8y = 7x + 2 \Rightarrow 7x - 8y = -2$

Step 2: Write the equations in matrix form.

coefficients of first equation → ; coefficients of second equation →

$$\begin{pmatrix} 2 & -2 \\ 7 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

← constant of first equation ; ← constant of second equation

Step 3: Find the inverse of the 2 × 2 matrix.

Determinant = (2 × −8) − (−2 × 7) = − 2

$$\text{Inverse} = -\frac{1}{2}\begin{pmatrix} -8 & 2 \\ -7 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 3.5 & -1 \end{pmatrix}$$

Step 4: Multiply both sides of the matrix equations with the inverse

$$\begin{pmatrix}4 & -1\\ 3.5 & -1\end{pmatrix}\begin{pmatrix}2 & -2\\ 7 & -8\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}4 & -1\\ 3.5 & -1\end{pmatrix}\begin{pmatrix}3\\ -2\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}14\\ 12.5\end{pmatrix}$$

So, $x = 14$ and $y = 12.5$

Example 3

Solve the simultaneous equations below using the matrix method

$x + 2y = 4$

$x + y = 3$

Solution

$$\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}4\\ 3\end{pmatrix}$$

Let $A = \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}, B = \begin{pmatrix}x\\ y\end{pmatrix}$ and $C = \begin{pmatrix}4\\ 3\end{pmatrix}$

Now $AB = C \Rightarrow B = \frac{C}{A}$

$B = A^{-1}C$

$\text{Det } A = (1 \times 1) - (2 \times 1) = -1$

$$A^{-1} = \frac{1}{-1}\begin{pmatrix}1 & -2\\ -1 & 1\end{pmatrix} = \begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}$$

But from $B = A^{-1}C$

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}4\\ 3\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-1\times 4 + 2\times 3\\ 1\times 4 + -1\times 3\end{pmatrix} = \begin{pmatrix}-4+6\\ 4+-3\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix}$$

From equality of matrices $x = 2$ and $y = 1$

Example 4

Solve the simultaneous equations using the matrix method

$2x + y = 3$

$4x - 2y = 10$

Solution

$$\begin{pmatrix}2 & 1\\ 4 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3\\ 10\end{pmatrix}$$

Let $A = \begin{pmatrix}2 & 1\\ 4 & -2\end{pmatrix}$

$\text{Det } A = (2 \times -2) - (1 \times 4) = -8$

$$A^{-1} = \frac{1}{\det A} \times Adjoint\ A$$

$$= \frac{1}{-8}\begin{pmatrix}-2 & -1\\ -4 & 2\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix} = -\frac{1}{8}\begin{pmatrix}-2 & -1\\ -4 & 2\end{pmatrix}\begin{pmatrix}3\\ 10\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix} = -\frac{1}{8}\begin{pmatrix}-2\times 3 + -1\times 10\\ -4\times 3 + 2\times 10\end{pmatrix} = -\frac{1}{8}\begin{pmatrix}-16\\ 8\end{pmatrix} = \begin{pmatrix}2\\ -1\end{pmatrix}$$

$x = 2$ and $y = -1$

Trial questions

1. Solve the following sets of simultaneous equations using the inverse matrix method.

a) $5x + y = 13$

$3x + 2y = 5$

b) $3x + 2y = -2$
$x + 4y = 6$

c) 4x + 2y = 6
3x + 5y =5

d) 7x + 4 =5y
4 – 2x + y =0

[Ans: a) $x = 3, y = -2,$ b) $x = -2, y = 2$ c) x = 10/7, y= 1/7 d) x= 8, y =12]

2. Given the matrices $A = \begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix}$ $and\ B = \begin{pmatrix} 6 & -1 \\ -2 & -3 \end{pmatrix}$, find;

(i) Matrix C which is equal to 2A – 3B

(ii) AB

(iii) Show that Det (A.B) = (Det A) (Det B) [Ans: (i) $\begin{pmatrix} -16 & 3 \\ 14 & 9 \end{pmatrix}$ (ii) $\begin{pmatrix} 6 & -1 \\ 14 & -19 \end{pmatrix}$]

3. Given that A = $\begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}$, determine (i) A + B (ii) $(AB)^2$

[Ans: (i) $\begin{pmatrix} 4 & 2 \\ 6 & -2 \end{pmatrix}$ (ii) $\begin{pmatrix} -45 & 30 \\ -45 & -50 \end{pmatrix}$]

5. Given that $\begin{pmatrix} 3-a & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -3 \\ x \end{pmatrix} = \begin{pmatrix} -3 \\ x \end{pmatrix}$, Find the values of $a\ and\ x$ [Ans: $a = 1, x = 1$]

6. Given the matrix $m = \begin{pmatrix} 3a & a-6 \\ -6 & a+2 \end{pmatrix}$, find the values of a for which the matrix m is singular

[Ans: a = -5.61, 1.61]

7. Given that $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}; B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, find A B – BA [Ans: $\begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$]

8. A and B are two matrices such that $A = \begin{pmatrix} 1 & 3 \\ 4 & 11 \end{pmatrix}$ and B = $\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$,Find (i) matrix P =AB (ii) P^{-1}

[Ans : (i) $\begin{pmatrix} 2 & 11 \\ 7 & 41 \end{pmatrix}$ (ii) $-\frac{1}{5}\begin{pmatrix} 41 & -11 \\ -7 & 2 \end{pmatrix}$]

9. Given the matrices P = $\begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}$ $and\ R = \begin{pmatrix} 5 & -4 \\ -1 & 2 \end{pmatrix}$; determine

(i) P.Q + R (ii) the determinant (P .Q + R) [Ans: (i) $\begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix}$ $(ii) - 3$]

9. Find the inverse of $A = \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}$ [Ans: $\frac{1}{14}\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$]

10. Given that $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, show that $A^2 - 4A = I$ where I is a 2 × 2 identity matrix.

11. Given that matrix A = $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$, find the values of the scalar λ for which $A - \lambda I$ where I is a 2 × 2 identity matrix. [Ans:$\lambda = 1\ or\ 4$]

DESCRIPTIVE STATISTICS

This is the branch of mathematics dealing with collection, interpretation, presentation and analysis of data where data refers to the facts in the day-to-day life.

Statistical data can be categorized into two .i.e. Qualitative and Quantitative.

Qualitative data measures attributes such as sex, colour, and so on while Quantitative data can be represented by numerical quantity. Quantitative data is of two forms. i.e. Continuous or discrete.

Discrete data is the information collected by counting and usually takes on integral values e.g. number of students in a class, school etc.

Continuous data can take on any value i.e. weight, height, mass, etc.

The quantity which is counted or measured is called the variable.

Crude/raw/ungrouped data

These are individual values of a variable in no particular order of magnitude, written down as they occurred or were measured.

Grouped /classified data

These are individual values of a variable that have been arranged in order and grouped in small number of classes.

Population and samples

A population is a total set of Items under consideration and its defined by some characteristics of these items.

A sample is a finite subset of a population.

PRESENTATION OF DATA

The ways of presenting data include:

- Bar graphs
- Histogram
- Frequency Polygon
- The Ogive
- Pie chart

BAR GRAPH

A bar graph or bar chart is a graph where the class frequencies are plotted against class limits.

HISTOGRAM

A histogram is a graph where the class frequencies are plotted versus class boundaries.

Example 1

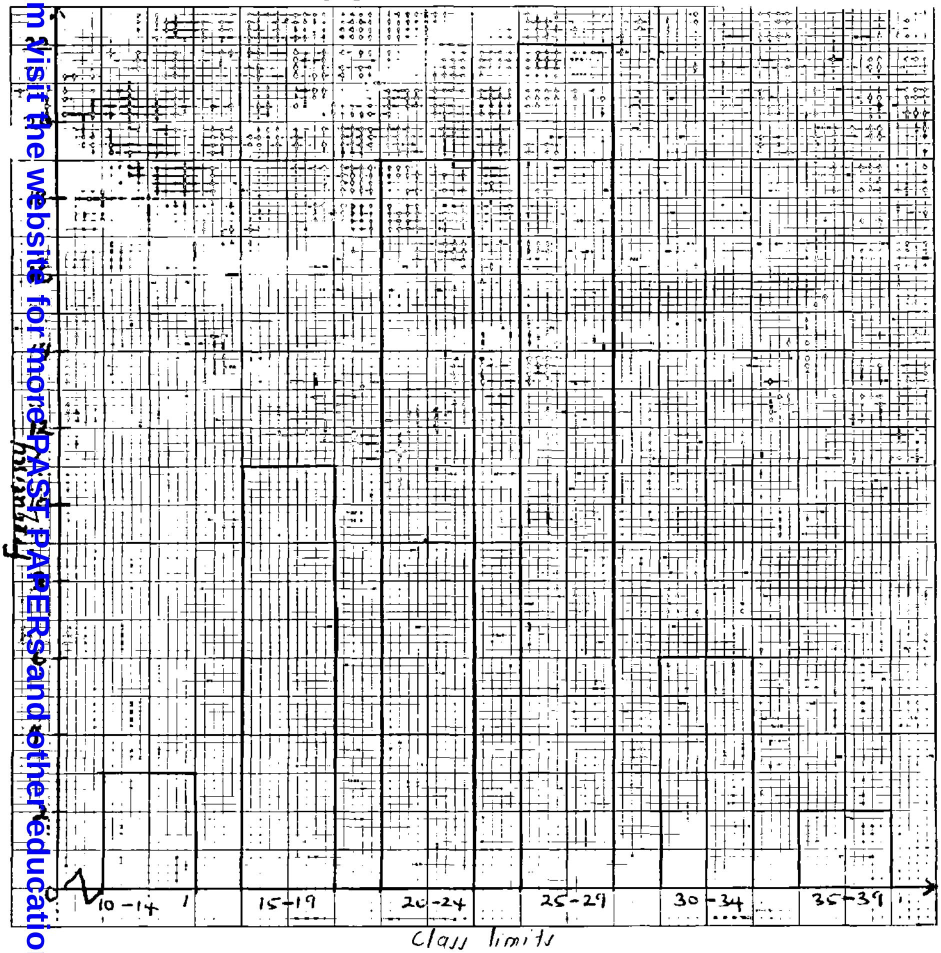
The times taken by rats to pass through a maze are recorded in the table below. Use the data given to plot a bar graph and histogram.

Time(seconds)	10-14	15-19	20-24	25-29	30-34	35-39
Frequency	3	11	19	22	6	2

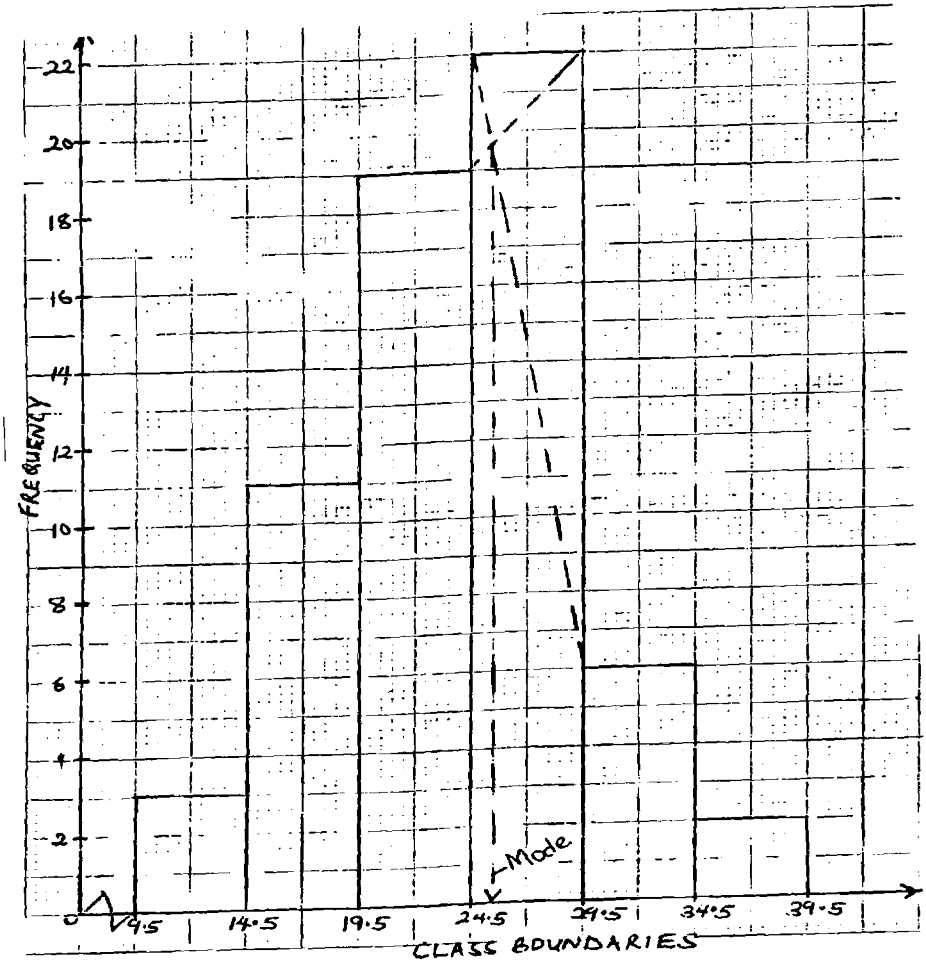
Solution

Class limits	Class boundaries	Frequency
10-14	9.5-14.5	3
15-19	14.5-19.5	11
20-24	19.5-24.5	19
25-29	24.5-29.5	22
30-34	29.5-34.5	6
35-39	34.5-39.5	2

Bar graph



<u>Histogram</u>



Note: The mode can be estimated from the histogram as shown above.
The reader should also note that these are spaces between the bars for a bar graph while there are no spaces for a histogram.

<u>Example 2</u>

The table below shows the population of Kampala in millions for different age groups

Age group	Population in millions
Below 10	2
10 and under 20	8
20 and under 30	10
30 and under 40	14
40 and under 50	5
50 and under 60	1

Draw a histogram to represent the above data

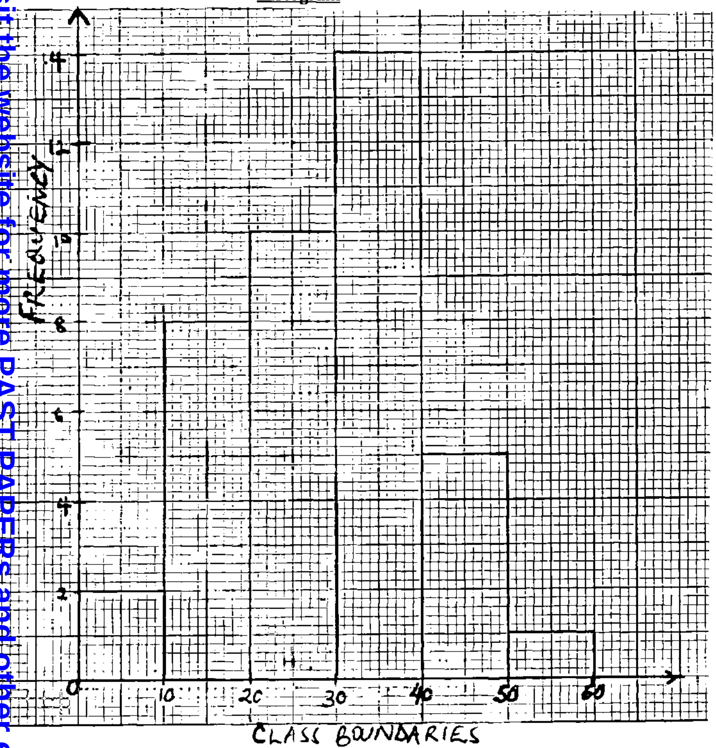
Solution

Class	Frequency
0-<10	2
10-<20	8
20-<30	10
30-<40	14
40-<50	5
50-<60	1

In this case,

The class boundaries are given i.e. 0-<10

Histogram



FREQUENCY POLYGON

The frequency polygon is obtained by plotting class frequencies versus class marks. Then the consecutive points are joined using a straight line.

Class mark/ mid interval value (x) = $\frac{1}{2}$ (Lower class limit + upper class limit)

i.e. for the class 10-14, class mark(x) =$\frac{1}{2}$(10+14)=12

The class mark is also known as the mid mark

Example 3

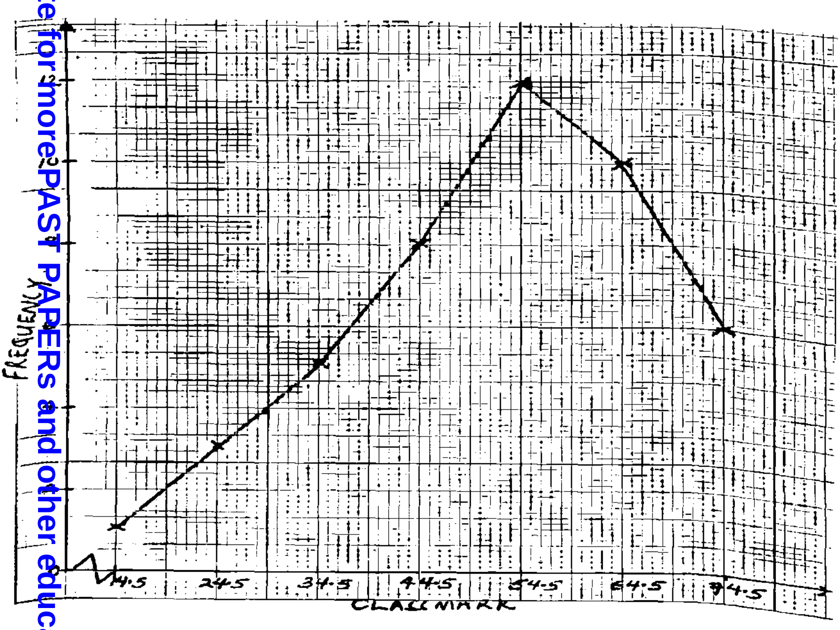
The age distribution of a group of people is given in the table below.

Age (year)	10-19	20-29	30-39	40-49	50-59	60-69	70-79
Frequency	1	3	5	8	12	10	6

Construct a frequency polygon for the data above

Solution

Class Limits	Class mark	Frequency
10-19	14.5	1
20-29	24.5	3
30-39	34.5	5
40-49	44.5	8
50-59	54.5	12
60-69	64.5	10
70-79	74.5	6



MEASURES OF CENTRAL TENDENCY

The measures of central tendency include the mean, mode and median. They are called so because they are centered about the same value.

MEAN

This is the sum of the data values divided by the number of values in the data.it is denoted by$\bar{X}$.

Mean,$\bar{X} = \frac{\sum x}{n}$ where $\sum$ means summation

The mean can also be calculated from ;

i) $\bar{X} = \frac{\sum fx}{\sum f}$

ii) $\bar{X} = A + \frac{\sum fd}{\sum f}$ Where A is the assumed/working mean and d = X—A where d is the deviation.

Examples

1. The measured weight for a child over eight year period gave the following results (in kgs); 32, 33, 35, 38, 43, 53, 63, 65. Calculate the mean weight of the child.

$$\text{Mean} = \frac{32+33+35+38+43+53+63+63+65}{8}$$

$$= 45.25\text{kg}$$

2. The information below gives the age in years of 49 students. Determine the mean age.

Age	14	15	16	17	18	21
Frequency	2	6	14	10	9	8

Solution

Age(x)	Frequency(f)	fx
14	2	28
15	6	90
16	14	224
17	10	170
18	9	162
21	8	168
	$\sum f = 49$	$\sum fx = 842$

$$\bar{X} = \frac{\sum fx}{\sum f} = \frac{842}{49} = 17.184 years$$

3. The data below shows the weights in kg of an S.5 class in a certain school.

Weight(kg)	10-14	15-19	20-24	25-29	30-34	35-39	40-44	45-49
Frequency	5	9	12	18	25	15	10	6

Calculate the mean weight of the class

Solution

Class	Class mark(x)	Frequency(f)	fx
10-14	12	5	60
15-19	17	9	153
20-24	22	12	264
25-29	27	18	486
30-34	32	25	800
35-39	37	15	555
40-44	42	10	420
45-49	47	6	282

Mean from assumed mean

The height to the nearest class of 30 pupils is shown in the table below. Using 152cm as the assumed mean, calculate the mean height.

Height, x(cm)	148	149	150	151	152	153	154	155	156
No. of Pupils	1	2	2	3	6	7	4	3	2

Solution

Assumed mean =152

Height(x)	Frequency(f)	Deviation(d=x-A)	fd
148	1	-4	-4
149	2	-3	-6
150	2	-2	-4
141	3	-1	-3
152	6	0	0
153	7	1	7
154	4	2	8
155	3	3	9
156	2	4	8
	$\sum f = 30$		$\sum fd = 15$

Mean, $\bar{X} = A + \frac{\sum fd}{\sum f}$

$\bar{X} = 152 + \frac{15}{30} = 152 + 0.5 = 152.5cm$

4. The number of accidents that took place at black spot on a certain road in 2008 were recorded as follows:

No. of accidents	0-4	5-7	8-10	11-13	14-18
No. of days	2	5	10	8	5

Using 9 as the working mean, calculate the mean no. of accidents per day.

Solution

Class	Mid value(x)	Freq(f)	Deviation(d)	fd
0-4	2	2	-7	-14
5-7	6	5	-3	-15
8-10	9	10	0	0
11-13	12	8	3	24
14-18	16	5	7	35
		∑f=30		∑fd=30

$$\text{Mean, } \bar{X} = A + \frac{\sum fd}{\sum f}$$

$$\bar{X} = 9 + \frac{30}{30} = 10$$

MEDIAN

The median of a group of numbers is the number in the middle when the numbers are in order of magnitude.

Determine the median for the following observations

(i) 4,1,6,2,6,7,8

Solution

1,2,4,6,6,7,8

The median is 6

(ii) 3,3,3,7,7,6,7,8

Solution

3, 3, 3, 4, 4, 6, 6, 7, 7

The median $= \frac{4+6}{2} = 5$

The formula below is used to obtain the median for grouped data.

$$\text{Median} = L_1 + \left(\frac{\frac{N}{2} - F_b}{f_m}\right) \times C$$

Where,

L_1 = lower class boundary of the median class

N = Total number of observations

F_b = Cumulative frequency before median class

f_m = Frequency of the median class

C = Class width

Class width is the difference between the lower and upper class boundaries ie for the class 40 – 44, the class width is 44.5 – 39.5=5

Note that it depends on the degree of accuracy ie for the class 7.0 – 7.4, the class width will be 7.45 – 6.95=0.5

Advantages of the median

It is easy to understand and calculate

It is not affected by extreme values

Disadvantage

It is only one or two values to decide the median

THE MODE

This is the number in a set of numbers that occurs the most i.e. the modal value of 5, 6, 3, 4, 5 2, 5 and 3 is 5 because there are more 5s than any other number.

For grouped data, the mode is calculated from;

$$\text{Mode} = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) \times C$$

Where;

L_1 = lower class boundary of the modal class

Δ_1 =difference between the modal frequency and the value before it

Δ_2 =difference between the modal frequency and the value after it

C = *class width*

The modal class is identified as the class with the highest frequency and the mode can as well be estimated from the histogram as we have already seen.

Example

The following were the heights of people in a certain town of Uganda.

Height(cm)	101-120	121-130	131-140	141-150	151-160	161-170	171-190
No. of p'ple	1	3	5	7	4	2	1

Calculate the mean, mode, and median for the data.

Solution

Class	Frequency(f)	Class mark(x)	fx	Cf	Class boundaries
101-120	1	110.5	110.5	1	100.5-120.5
121-130	3	125.5	376.5	4	120.5-130.5
131-140	5	135.5	677.5	9	130.5-140.5
141-150	7	145.5	1018.5	16	140.5-150.5
151-160	4	155.5	622	20	150.5-160.5
161-170	2	165.5	331	22	160.5-170.5
171-190	1	180.5	180.5	23	170.5-190.5
$\sum$	23		3316.5		

Mean, $\bar{X} = \frac{\sum fx}{\sum f} = \frac{3316.5}{23} = 144cm$

Median class is 141 – 150

$$\text{Median}=L_1 + \left(\frac{\frac{N}{2}-F_b}{fm}\right) \times C = 140.5 + \left(\frac{\frac{23}{2}-9}{7}\right) \times 10$$

$$= 140.5+3.57 = 144.1\text{cm}$$

$$\text{Mode} = L_1 + \left(\frac{\Delta_1}{\Delta_1+\Delta_2}\right) \times C$$

Modal class is 141 – 150

$\Delta_1 = 7 - 5 = 2$ *and* $\Delta_1 = 7 - 4 = 3$

$$\text{Mode} = 140.5+ \left(\frac{2}{2+3}\right) \times 10 = 140.5 + 4 = 144.5cm$$

Example

Using the data for example 3 (pg. 107), Calculate the mode and median.

Class	Freq(f)	Cf	Class boundaries
10-14	5	5	9.5-14.5
15-19	9	14	14.5-19.5
20-24	12	26	19.5-24.5
25-29	18	44	24.5-29.5
30-34	25	69	29.5-34.5
35-39	15	84	34.5-39.5
40-44	10	94	39.5-44.5
45-49	6	100	44.5-49.5

Median$=L_1+\left(\frac{\frac{N}{2}-F_b}{f_m}\right)\times C$

Median class is 30 – 34

Median$=29.5+\left(\frac{\frac{100}{2}-44}{25}\right)\times 5=29.5+1.2=30.7kg$

Mode $=L_1+\left(\frac{\Delta_1}{\Delta_1+\Delta_2}\right)\times C$

Modal class is 30 – 34, $\Delta_1=25-18=7 \quad and\ \Delta_1=25-15=10$

Mode = $29.5+\left(\frac{7}{7+10}\right)\times 5=29.5+2.06=31.56kg$

THE OGIVE

The Ogive is also known as the cumulative frequency curve where by cumulative frequency curve is plotted against the upper class boundaries and the consecutive points are joined into a smooth curve using free hand.

Example

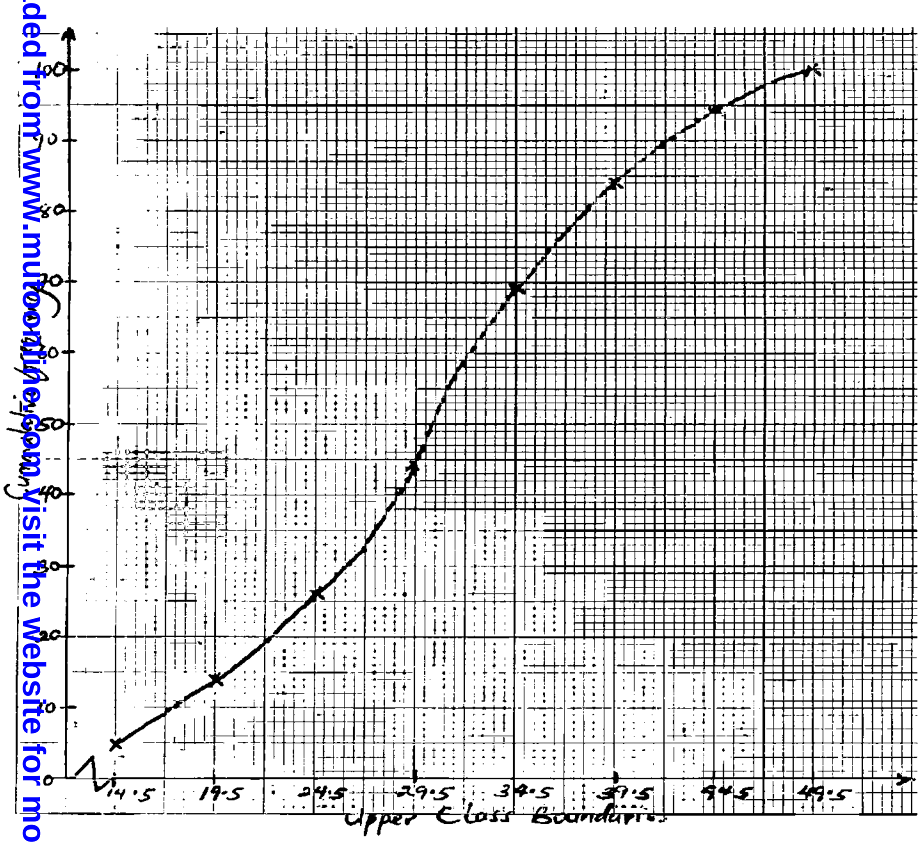
The frequency distribution table shows the weights of 100 children measured to the nearest kg.

Weight	10-14	15-19	20-24	25-29	30-34	35-39	40-44	45-39
No. of Children	5	9	12	18	25	15	10	6

Draw a cumulative frequency curve for the data.

Solution

Class	Freq(f)	Cf	Class boundary
10-14	5	5	9.5-14.5
15-19	9	14	14.5-19.5
20-24	12	26	19.5-24.5
25-29	18	44	24.5-29.5
30-34	25	69	29.5-34.5
35-39	15	84	34.5-39.5
40-44	10	94	39.5-44.5
45-49	6	100	44.5-49.5



Estimating the median and quartiles using the Ogive.

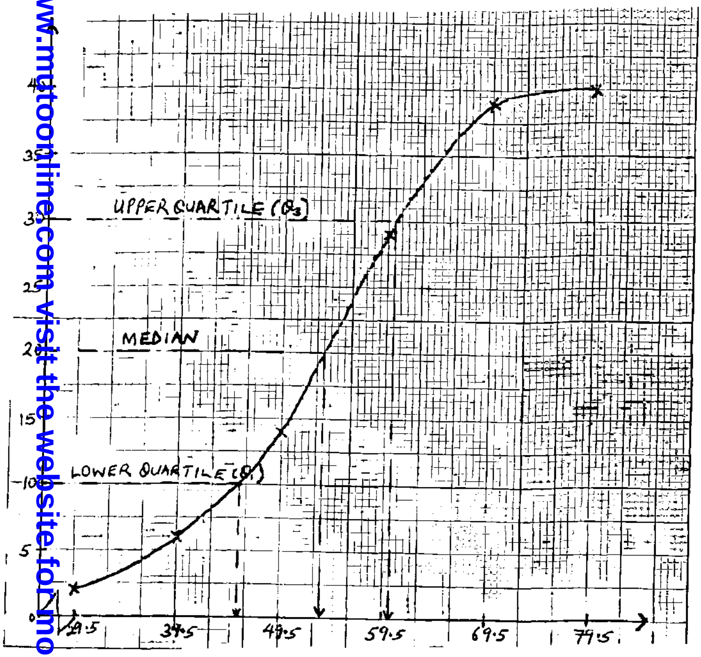
The marks obtained by 40 pupils in a mathematics examination were as follows:

Marks	20-29	30-39	40-49	50-59	60-69	70-79
No. of pupils	2	4	8	15	9	2

Plot a cumulative frequency curve and use it to estimate the median mark, upper quartile, lower quartile and the inter quartile range

Solution

Class	Freq(f)	Cf	Upper class boundaries
20-29	2	2	29.5
30-39	4	6	39.5
40-49	8	14	49.5
50-59	15	29	59.5
60-69	9	38	69.5
70-79	2	40	79.5



Upper class boundaries

Median $=\left(\frac{1}{2}N\right)^{th}$ =20th measure

Draw a dotted line across the graph from Cf = 20 to meet the curve and drop a vertical dotted line to meet the horizontal axis. This gives the estimated median

Hence the median= 54 marks.

Quartiles

The quartiles divide a distribution into four equal parts.

The lower quartile (Q_1) is the value 25% way through the distribution and the value 75% way through the distribution is called the upper quartile (Q_3).

Lower quartile (Q_1) $= \left(\frac{1}{4}N\right)^{th}$ $measure$ $= 45.5$

Upper quartile, (Q_3) $= \left(\frac{3}{4}N\right)^{th}$ $measure$ $=60$

The difference between the upper quartile and lower quartile is called the Interquartile range.The Interquartile range = $Q_3 - Q_1$ = 60 – 45.5 =14.5

The semi interquartile range or quartile deviation$=\frac{1}{2}(Q_3 - Q_1) = 7.25$

Percentiles

The percentiles divide a distribution into one hundred equal parts.

The lower quartile, Q_1 is the 25th percentile P25, the median is the 50th percentile P50 and the upper quartile Q_3 is the 75th percentile P75.

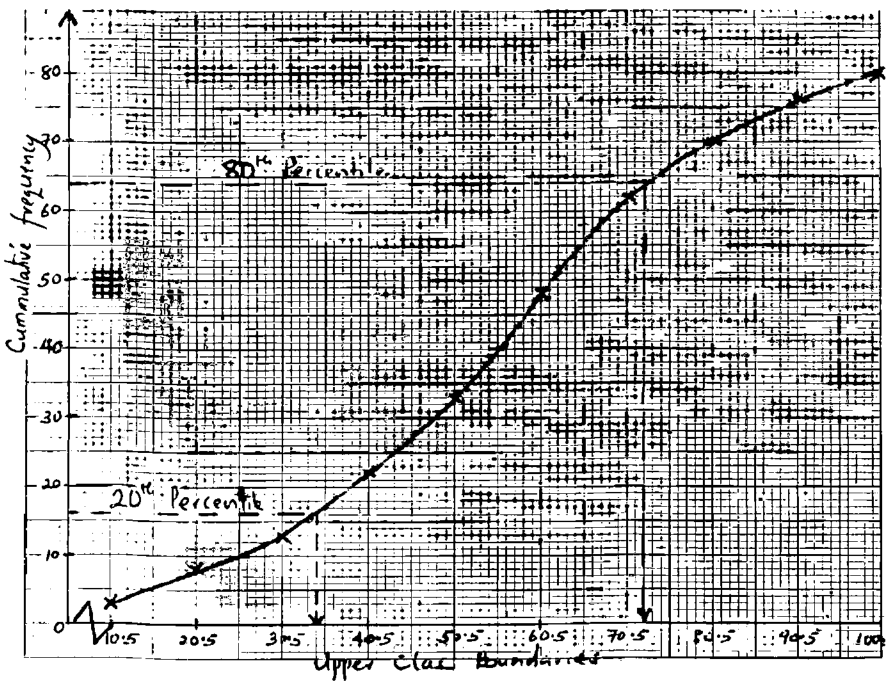
Example

The data shows the marks obtained by 80 form IV pupils in a school. Draw a cumulative frequency and use your graph to find the 20th and 80th percentile mark.

Mark	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Freq	3	5	5	9	11	15	14	8	6	4

Solution

Marks	Freq	C.f	Upper class boundaries
1-10	3	3	10.5
11-20	5	8	20.5
21-30	5	13	30.5
31-40	9	22	40.5
41-50	11	33	50.5
51-60	15	48	60.5
61-70	14	62	70.5
71-80	8	70	80.5
81-90	6	76	90.5
91-100	4	80	100.5



20th percentile mark=$\left(\frac{20}{100}\times 80\right)^{th}$ $mark = 32.5$

80th percentile mark=$\left(\frac{80}{100}\times 80\right)^{th}$ $mark$ =71.5

Measures of dispersion

The spread of observations in relation to a measure of central tendency of the given data is known as dispersion. In order to compare data, the measure of dispersion is taken into account along with the measure of central tendency.

The range

This is the difference between the largest and the smallest values of the data.
i.e. for the data about lengths of leaves in garden tree, 5,6,7,7,4,5,3,2,9,8,8,6,5,3
Range = 9-2 =7

Standard deviation:

This is the positive square root of variance. It is denoted by σ
Standard deviation $(\sigma) = \sqrt{Variance}$
The following expressions can be used to calculate the standard deviation;

$$\sigma = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2}$$

When using the assumed mean A,

$$\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$$

Note: the expression under the root is the variance

Examples

1. Calculate the standard deviation for the distribution of marks in the table below.

Marks	5	6	7	8	9
Frequency	3	8	9	6	4

Solution

Marks(x)	Frequency(f)	fx	fx²
5	3	15	75
6	8	48	288
7	9	63	441
8	6	48	384
9	4	36	324
Σ	30	210	1512

$$\sigma = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2}$$

$$\sigma = \sqrt{\frac{1512}{30} - \left(\frac{210}{30}\right)^2} = \sqrt{50.4 - 49} = \sqrt{1.4} = 1.183\ marks$$

2. The table below shows the weights to the nearest kg of 150 patients who visited a certain health unit during a certain week.

Weight(kg)	10-19	20-29	30-39	40-49	50-59	60-69	70-79
No. of patients	30	16	24	32	28	12	8

Calculate the standard deviation of the weights of the patients.

Class	Freq(f)	x	fx	fx²
10-19	30	14.5	435	6307.5
20-29	16	24.5	392	9604
30-39	24	34.5	828	28566
40-49	32	44.5	1424	63368
50-59	28	54.5	1526	83167
60-69	12	64.5	774	49923
70-79	8	74.5	596	44402
	∑f= 150		∑fx=5975	∑fx² =285337.5

Standard deviation, $\sigma = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2}$

$$\sigma = \sqrt{\frac{285337.5}{150} - \left(\frac{5975}{150}\right)^2}$$

$$=\sqrt{315.56} = 17.76$$

3. The table below gives the points scored by a team in various events. Find the mean and standard deviation using working mean A=4

Points	0	1	2	3	4	5	6	7
No. of events	1	3	4	7	5	5	2	3

Solution

Points	Frequency	d= x-A	fd	fd²
0	1	-4	-4	16
1	3	-3	-9	27
2	4	-2	-8	16
3	7	-1	-7	7
4	5	0	0	0
5	5	1	5	5
6	2	2	4	8
7	3	3	9	27
Σ	30		-10	100

Mean, $\bar{X} = A + \frac{\Sigma fd}{\Sigma f} = 4 + \frac{-10}{30}$ =3.67points

Standard deviation, $\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$

$$= \sqrt{\frac{106}{30} - \left(\frac{-10}{30}\right)^2} = \sqrt{3.533 - 0.111} = 1.85 points$$

4. The table below shows the weight in kg of 100 boys in a certain school

Weight(kg)	60-62	63-65	66-68	69-71	72-74
Frequency	8	10	45	30	7

Using the assumed mean of 67, calculate the mean and standard deviation

Solution

Weight	Freq(f)	Mid value (x)	D	fd	fd^2
60-62	8	61	-6	-48	288
63-65	10	64	-3	-30	90
66-68	45	67	0	0	0
69-71	30	70	3	3	270
72-74	7	73	6	6	252
	$\sum f=100$			$\sum fd=54$	$\sum fd^2=900$

Mean, $\bar{X} = A + \frac{\sum fd}{\sum f} = 67 + \frac{54}{100} = 67.54kg$

Standard deviation, $\sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$

$$= \sqrt{\frac{900}{100} - \left(\frac{54}{100}\right)^2} = 2.951$$

Trial questions

1. The table below shows the weekly wages of a number of workers at a small factory.

Weekly wages	75-84	85-94	95-104	105-114	115-124	125-134	135-144	145-154
Frequency	2	3	7	11	10	8	4	1

Calculate the modal, median and the mean wage.

2. Below are heights, measured to the nearest cm of 50 pupils

157	167	165	162	160	157	160	152	157	162
157	165	152	162	155	160	157	160	162	160
157	152	167	157	160	160	162	165	157	160
157	157	157	160	157	162	155	157	160	157
150	162	152	160	157	157	165	160	162	150

a) Make a frequency distribution table by dividing them into class intervals of 5 starting with the class 148-152

b) Draw a cumulative frequency curve and use it to estimate

(i) The median (ii) Interquartile range

3. The table below shows marks obtained by students of mathematics in a certain school.

Marks	30-<40	40-<50	50-<60	60-<70	70-<80
No. of students	2	15	10	11	27

(i) Calculate the mean, median and standard deviation for the above data

(ii) Draw an Ogive for the above data

4. Below are heights, measured to the nearest cm of 50 pupils.

157	167	165	162	160	157	160	152	157	162
157	165	152	162	155	160	157	160	162	160
157	152	167	157	160	160	162	165	157	160
157	157	157	160	157	162	155	157	160	157
150	162	152	160	157	157	165	160	162	150

a) Make a frequency distribution table by dividing them into class intervals of 5 starting with the class 148-152

b) Draw a cumulative frequency curve and use it to estimate

(i) The median (ii) Interquartile range

5. The table below shows marks obtained by students of mathematics in a certain school

Marks	30-<40	40-<50	50-<60	60-<70	70-<80
No. of students	2	15	10	11	27

(i) Calculate the mean, median and standard deviation for the above data

(ii) Draw an Ogive for the above data

6. Sixty pupils were asked to draw a free hand line of length 20cm. The lengths of the lines were measured to nearest cm, and were recorded as shown in the table.

Length(cm)	11-13	13-15	15-17	17-19	19-21	21-23	23-25
Frequency	3	6	11	15	13	10	2

a) Calculate the mean length

b) Draw a cumulative frequency graph and estimate the median, the upper and the lower quartiles.

7. Below are the heights to the nearest cm of 40 students

150	170	152	155	169	167	157	158	157
167	164	165	164	163	162	163	158	158
160	160	159	161	161	161	160	160	160
159	162	160	159	160	161	161	156	150

a) Make a frequency distribution table starting with class interval 150-152

b) Draw an Ogive and use it to estimate the median, Interquartile range and the 20^{th} percentile height.

8. Calculate the mean and the standard deviation of the following distribution of scores

Scores	1-5	6-10	11-15	16-20	21-25	26-30	31-35
Frequency	3	19	38	69	45	21	5

9. The numbers of the eggs collected from a poultry farm for 40 consecutive days were as follows.

138	145	145	157	150	142	154	140
146	135	128	149	164	147	152	138
168	142	135	125	158	135	148	176
146	150	165	144	126	153	136	163
161	156	144	132	176	140	147	130

a) Construct a frequency distribution table with classes of equal interval width 5, starting from 125-129.

b) Draw a cumulative frequency curve (Ogive) and use it to estimate the

(i) Median

(ii) Interquartile range

(iii) Median number of eggs

10. The following marks were obtained by 85 students in an English examination;

96 81 23 62 44 18 62 70 72 40 81 70 30 28 23 02
60 20 48 50 19 33 32 58 71 62 19 12 83 53 81 73
52 25 71 61 46 64 35 59 82 82 42 63 43 17 35 72
37 54 47 76 18 44 65 45 70 38 63 89 31 37 93 03
63 25 52 53 38 57 53 71 70 63 89 31 37 93 58 58

a) Using class intervals of 10 marks, and starting with a class of 0-9, construct a frequency distribution table.

b) Using your table to find the (i) Median mark

(ii) Mean mark

(iii) Standard deviation

11. The marks obtained by 50 students in a test were:

76 17 57 63 12 96 38 46 82 48
61 93 44 19 70 60 71 18 40 54
50 27 62 42 63 52 53 38 62 25
62 23 32 81 31 63 64 18 70 27
52 81 35 63 38 37 44 19 70 32

a) Construct a grouped frequency distribution table with equal class intervals of 10 marks, starting with the 10 – 19 class group.

b) Draw a histogram and use it to estimate the modal mark.

c) Calculate the mean and standard deviation of the mark.

12. The times taken by a group of students to solve a mathematical problem are given below.

Time(min)	5-9	10-14	15-19	20-24	25-29	30-34
No. of students	5	14	30	17	11	3

a) Draw a histogram for the data. Use it to estimate the modal time for solving a problem.
b) Calculate the mean time and standard deviation of solving a problem.

13. The table below shows the weights (in kg) of 150 patients who visited a certain health unit during a certain week.

Weight (kg)	0-19	20-29	30-39	40-49	50-59	60-69	70-79
No. of patients	30	16	24	32	28	12	8

a) Calculate the appropriate mean and modal weights of the patients.
b) Plot an Ogive for the above data. Use the Ogive to estimate the median and semi interquartile for the weights of patients.

14. In agricultural experiment, the gains in mass (in kg) of 100 cows during a certain period were recorded as follows;

Gain in mass (kgs)	5 - 9	10-14	15-19	20-24	25-29	30-34
Frequency	2	29	37	16	14	2

a) Calculate the (i)mean mass gained
(ii)Standard deviation
(iii)Median

15. The information below shows the marks of 36 candidates in oral examination.

30 31 55 49 56 47
36 41 39 45 39 50
42 43 44 39 46 56
30 48 53 38 50 63
40 54 61 46 56 44
53 60 56 50 62 52

(i) Construct a frequency distribution table having an interval of 6marks starting with the 30-35 class group.
(ii) Draw a cumulative frequency curve and use it to estimate the median mark.
(iii) Calculate the mean mark.

16. Construct a frequency distribution of the following data on the length 5 of time (in minutes), it took 50 persons to complete a certain application form.

29 22 38 28 34 32 23 19 21 31
16 28 19 18 12 27 15 21 25 16
30 17 22 29 18 29 25 20 16 11
17 12 15 24 25 21 22 17 18 15
21 20 23 18 17 15 16 26 23 22

Using class intervals of length 5minutes starting with the interval 10-14. Calculate the (i) Mean (ii) Standard deviation using ; Assumed mean A= 22

17. The ages of students in an Institution were as follows.

Age	18-<19	19-<20	20-<21	21-<22	23-<24	24-<25
No. of students	12	35	38	24	8	3

(i) Draw a histogram of the data and use it to estimate the modal age.

(ii) Use the data to estimate the median, upper and lower quartile ages.

(iii) Calculate the interquartile and semi interquartile range

18. Estimate the lower and upper quartiles for the following frequency distribution using an Ogive.

Class	0-9	10-19	20-29	30-39	40-49
Frequency	2	14	24	12	8

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